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OPTIMAL DAMPER PLACEMENT FOR MINIMUM TRANSFER FUNCTIONS

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SUMMARY

The purpose of this paper is to propose an efficient and systematic procedure for finding the optimal damper placement to minimize the sum of amplitudes of the transfer functions evaluated at the undamped fundamental natural frequency of a structural system subject to a constraint on the sum of the damping coefficients of added dampers. Optimality criteria are derived and the optimal damper placement is determined based upon those criteria without any indefinite iterative operation. The present procedure can be applied to any structural system so far as it can be modelled with finite-element systems. The present procedure also enables one to treat structural systems with an arbitrary damping system (for example, proportional or non-proportional) in a unified manner. Due to employment of a general dynamical property, i.e. the amplitude of a transfer function, the results are general and are not influenced by characteristics of input motions.

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KEY WORDS: optimal damper placement; passive control; transfer function; non-proportional damping; design sensitivity analysis

1. INTRODUCTION

The problem treated in the present paper is to find the optimal damper placement to minimize the sum of amplitudes of the transfer functions evaluated at the undamped fundamental natural frequency of a structural system subject to a constraint on the sum of the damping coefficients of added dampers. This problem is a kind of inverse problems. Extensive studies have ever been performed in the field of inverse eigenmode problems of updamped systems.[1–4] However those results may not necessarily be applicable to the inverse problem of damped systems. In particular, when the magnitude of damping is fairly large or the damping system is non-classical (non-proportional), the inverse problems for undamped systems do not necessarily provide relevant informations on design of the corresponding damped systems.

In the present paper an efficient and systematic algorithm for the optimal damper placement is proposed for structural systems with an arbitrary damping system (for example, proportional or non-proportional). The features of the present formulation are to be able to deal with any damping system, e.g. proportional or non-proportional, to be able to treat any structural system so far as it can be modelled with finite-element systems and to consist of a systematic algorithm without any indefinite iterative operation. It should also be

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pointed out that, because the present paper deals with a general dynamical property, i.e. the amplitude of the transfer function, the results are general and are not influenced by characteristics of input motions.

Research on optimal passive damper placement has been very limited. The following studies may be relevant to the present paper. Constantinou and Tadjbakhsh[5] derived the optimum damping coefficient for a damper placed on the first storey of a shear building subjected to horizontal random earthquake motions. Gürgöze and Müller[6] presented a numerical method for finding the optimal placement and the optimal damping coefficient for a single viscous damper in a prescribed linear multi-degree-of-freedom system. Zhang and Soong[7] proposed a seismic design method to find the optimal configuration of viscous dampers for a building with specified storey stiffnesses. While their method is based upon an intuitive criterion that an additional damper should be placed sequentially on the storey with the maximum interstorey drift, it is pioneering. Hahn and Sathiavageeswaran[8] performed several parametric studies on the effects of damper distribution on the earthquake response of shear buildings, and showed that, for a building with uniform storey stiffnesses, dampers should be added to the lower half floors of the building. De Silva[9] presented a gradient algorithm for the optimal design of discrete passive dampers in the vibration control of a class of flexible systems. Inaudi and Kelly[10] proposed a procedure for finding the optimal isolation damping for minimum acceleration response of base-isolated structures subjected to stationary random excitation. Tsuji and Nakamura[11] proposed an algorithm to find both the optimal storey stiffness distribution and the optimal damper distribution for a shear building model subjected to a set of spectrum-compatible earthquakes. Masri *et al.*[12] presented a simple yet efficient optimum active control method for reducing the oscillations of distributed parameter systems subjected to arbitrary deterministic or stochastic excitations. While they deal with active control, the result is informative to the development in passive optimal control theories.

The following studies on control of amplitudes of transfer functions also appear to be relevant to the present paper. Tsai[13] proposed a technique to determine approximate modal damping ratios as a classically damped model for a non-classically damped soil–structure system by matching amplitudes of transfer functions at several natural frequencies in both models and solved a set of simultaneous *nonlinear* equations iteratively. Chen[14] formulated an optimal design problem for a *classically damped* model subject to constraints on natural frequencies and frequency responses at a fixed frequency.

2. PROBLEM OF OPTIMAL DAMPER PLACEMENT FOR A SHEAR BUILDING MODEL

2.1. Problem of optimal damper placement

Consider as an illustrative example a two-storey shear building model with added viscous dampers as shown in Figure 1. Let $\{\bar{m}_1, \bar{m}_2\}$ and $\{\bar{k}_1, \bar{k}_2\}$ denote the masses and storey stiffnesses of the shear building model and let $\{c_1, c_2\}$ denote the damping coefficients of the added dampers. Assume that $\{\bar{m}_1, \bar{m}_2\}$ and

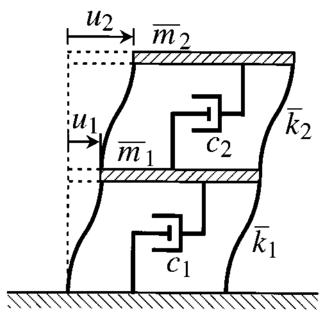

Figure 1. Two-storey shear building model with added viscous dampers

$\{\bar{k}_1, \bar{k}_2\}$ be prescribed. The design variables are $\mathbf{c} = \{c_1, c_2\}$. It is also assumed here that the original structural damping is negligible compared to the damping of the added dampers. Let u_1 and u_2 denote the displacements of masses $\bar{m}_1$ and $\bar{m}_2$, respectively. When this model is subjected to a base acceleration $\ddot{u}_g$, the equation of motion for this model is written as

$$\begin{bmatrix} \bar{k}_1 + \bar{k}_2 & -\bar{k}_2 \\ -\bar{k}_2 & \bar{k}_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} + \begin{bmatrix} \bar{m}_1 & 0 \\ 0 & \bar{m}_2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} = -\begin{bmatrix} \bar{m}_1 & 0 \\ 0 & \bar{m}_2 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \ddot{u}_g \quad (1)$$

Let $U_1(\omega)$, $U_2(\omega)$, $\ddot{U}_g(\omega)$ denote the Fourier transforms of $u_1, u_2, \ddot{u}_g$, respectively, and let ω denote a circular frequency of excitation. Fourier transformation of equation (1) may be reduced to the following form:

$$\left(\begin{bmatrix} \bar{k}_1 + \bar{k}_2 & -\bar{k}_2 \\ -\bar{k}_2 & \bar{k}_2 \end{bmatrix} + \mathrm{i}\omega \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} - \omega^2 \begin{bmatrix} \bar{m}_1 & 0 \\ 0 & \bar{m}_2 \end{bmatrix} \right) \begin{Bmatrix} U_1(\omega) \\ U_2(\omega) \end{Bmatrix} = -\begin{bmatrix} \bar{m}_1 & 0 \\ 0 & \bar{m}_2 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \ddot{U}_g(\omega) \quad (2)$$

where i is the imaginary unit. The Fourier transforms $\delta_1(\omega)$, $\delta_2(\omega)$ of the interstorey drifts $d_1 = u_1$, $d_2 = u_2 - u_1$ are related to $U_1(\omega)$, $U_2(\omega)$ by

$$\begin{Bmatrix} \delta_1(\omega) \\ \delta_2(\omega) \end{Bmatrix} = \mathbf{T} \begin{Bmatrix} U_1(\omega) \\ U_2(\omega) \end{Bmatrix} \quad (3)$$

where

$$\mathbf{T} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad (4)$$

Let ω_1 denote the undamped fundamental natural frequency of the model and let us define new quantities $\hat{U}_1$, $\hat{U}_2$ by

$$\hat{U}_1 \equiv U_1(\omega_1)/\ddot{U}_g(\omega_1), \ \hat{U}_2 \equiv U_2(\omega_1)/\ddot{U}_g(\omega_1). \quad (5)$$

$\hat{U}_i$ is equal to the value such that ω_1 is substituted in the frequency response function which can be obtained as $U_i(\omega)$ after substituting $\ddot{U}_g(\omega) = 1$ in equation (2). It should be noted that, because $\{\bar{m}_1, \bar{m}_2\}$ and $\{\bar{k}_1, \bar{k}_2\}$ are prescribed, ω_1 is a given value. Furthermore, new quantities $\hat{\delta}_1$, $\hat{\delta}_2$ are defined by $\hat{\delta}_1 \equiv \hat{U}_1$, $\hat{\delta}_2 \equiv \hat{U}_2 - \hat{U}_1$ (see Figure 2). Due to equations (2) ($\omega = \omega_1$) and (5), $\hat{U}_1$, $\hat{U}_2$ must satisfy the following

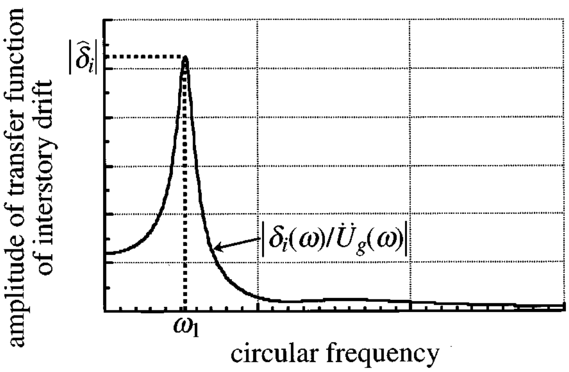

Figure 2. Amplitude of a transfer function of an interstorey drift with respect to excitation frequency and its value evaluated at the undamped fundamental natural frequency

equation:

$$\mathbf{A}\begin{Bmatrix}\hat{U}_1\\ \hat{U}_2\end{Bmatrix} = -\begin{Bmatrix}\bar{m}_2\\ \bar{m}_2\end{Bmatrix} \tag{6}$$

where

$$\mathbf{A} = \begin{bmatrix}(\bar{k}_1+\bar{k}_2)+\mathrm{i}\omega_1(c_1+c_2)-\omega_1^2\bar{m}_1 & -\bar{k}_2-\mathrm{i}\omega_1 c_2\\ -\bar{k}_2-\mathrm{i}\omega_1 c_2 & \bar{k}_2+\mathrm{i}\omega_1 c_2-\omega_1^2\bar{m}_2\end{bmatrix} \tag{7}$$

It should be remarked here that the squares of the amplitudes of transfer functions are meaningful from physical points of view because they can be transformed into response mean squares (statistical quantities) after multiplication with the power spectral density function of a disturbance and integration in the frequency range. Since the amplitude of the transfer function of an interstorey drift evaluated at the undamped fundamental natural circular frequency can be related to the level of this response mean square, these amplitudes of transfer functions are treated as controlled quantities in the present paper.

The Problem of Optimal Damper Placement may be described as follows:

Problem PODP: Find the damping coefficients of added dampers which minimize the sum of amplitudes of the transfer functions of interstorey drifts evaluated at the undamped fundamental natural frequency

$$V = \sum_{i=1}^{2} |\hat{\delta}_i(\mathbf{c})| \tag{8}$$

subject to a constraint on the sum of the damping coefficients of added dampers

$$\sum_{i=1}^{2} c_i = \bar{W} \quad (\bar{W}\text{: specified value}) \tag{9}$$

The Lagrangian L for Problem PODP may be expressed in terms of Lagrange multiplier λ.

$$L(\mathbf{c}, \lambda) = \sum_{i=1}^{2} |\hat{\delta}_i(\mathbf{c})| + \lambda\left(\sum_{i=1}^{2} c_i - \bar{W}\right) \tag{10}$$

For simplicity of expression the argument ($\mathbf{c}$) will be omitted hereafter.

2.2. *Optimality criteria*

The optimality criteria for Problem PODP can be derived from stationary conditions of $L(\mathbf{c}, \lambda)$ with respect to $\mathbf{c}$ and λ.

$$\left(\sum_{i=1}^{2} |\hat{\delta}_i|\right)_{,j} + \lambda = 0 \ (j = 1, 2) \tag{11}$$

$$\sum_{i=1}^{2} c_i - \bar{W} = 0 \tag{12}$$

Here and in the following $(\ )_{,j}$ denotes the partial differentiation with respect to c_j. It should be noted that the damping coefficient of every added damper must be a non-negative value. If $c_j = 0$, equation (11) must be modified into the following form:

$$\left(\sum_{i=1}^{2} |\hat{\delta}_i|\right)_{,j} + \lambda \geqslant 0 \tag{13}$$

The optimality criteria, equation (11), include a parameter λ. It may be convenient from a viewpoint of construction of a systematic solution algorithm to express the optimality criteria without this parameter. Let

us define a new quantity defined by

$$\gamma_1 = \left(\sum_{i=1}^{2} |\hat{\delta}_i|\right)_{,2} \Bigg/ \left(\sum_{i=1}^{2} |\hat{\delta}_i|\right)_{,1} \tag{14}$$

The alternative expression of the optimality criteria, equation (11), may then be reduced to $\gamma_1 = 1$. If $c_1 = 0$, $\gamma_1 \geqslant 1$. If $c_2 = 0$, $\gamma_1 \leqslant 1$.

2.3. Solution algorithm

Differentiation of equation (6) with respect to a design variable c_j provides

$$\mathbf{A}_{,j} \begin{Bmatrix} \hat{U}_1 \\ \hat{U}_2 \end{Bmatrix} + \mathbf{A} \begin{Bmatrix} \hat{U}_{1,j} \\ \hat{U}_{2,j} \end{Bmatrix} = \mathbf{0} \tag{15}$$

$\mathbf{A}_{,j}$ in equation (15) can be expressed as

$$\mathbf{A}_{,1} = \mathrm{i}\omega_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{A}_{,2} = \mathrm{i}\omega_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{16}$$

Since $\mathbf{A}$ is regular, the first-order sensitivities of $\hat{U}_1$, $\hat{U}_2$ are derived from equation (15) as

$$\begin{Bmatrix} \hat{U}_{1,j} \\ \hat{U}_{2,j} \end{Bmatrix} = -\mathbf{A}^{-1}\mathbf{A}_{,j} \begin{Bmatrix} \hat{U}_1 \\ \hat{U}_2 \end{Bmatrix} \tag{17}$$

The first-order sensitivities of $\hat{\delta}_1$, $\hat{\delta}_2$ are derived by substituting $\hat{\delta}_1 \equiv \hat{U}_1$, $\hat{\delta}_2 \equiv \hat{U}_2 - \hat{U}_1$ into equation (17) as

$$\begin{Bmatrix} \hat{\delta}_{1,j} \\ \hat{\delta}_{2,j} \end{Bmatrix} = -\mathbf{T}\mathbf{A}^{-1}\mathbf{A}_{,j}\mathbf{T}^{-1} \begin{Bmatrix} \hat{\delta}_1 \\ \hat{\delta}_2 \end{Bmatrix} \tag{18}$$

The quantity $\hat{\delta}_i$ may be rewritten as

$$\hat{\delta}_i = \mathrm{Re}[\hat{\delta}_i] + \mathrm{i}\,\mathrm{Im}[\hat{\delta}_i] \tag{19}$$

where Re[ ] and Im[ ] indicate the real and imaginary parts, respectively, of a complex number. The first-order sensitivity of $\hat{\delta}_i$ may be formally expressed as

$$\hat{\delta}_{i,j} = (\mathrm{Re}[\hat{\delta}_i])_{,j} + \mathrm{i}(\mathrm{Im}[\hat{\delta}_i])_{,j}. \tag{20}$$

The absolute value of $\hat{\delta}_i$ is defined by

$$|\hat{\delta}_i| = \sqrt{(\mathrm{Re}[\hat{\delta}_i])^2 + (\mathrm{Im}[\hat{\delta}_i])^2} \tag{21}$$

The first-order sensitivity of $|\hat{\delta}_i|$ may then be expressed as

$$|\hat{\delta}_i|_{,j} = \frac{1}{|\hat{\delta}_i|}\{\mathrm{Re}[\hat{\delta}_i](\mathrm{Re}[\hat{\delta}_i])_{,j} + \mathrm{Im}[\hat{\delta}_i](\mathrm{Im}[\hat{\delta}_i])_{,j}\} \tag{22}$$

where $(\mathrm{Re}[\hat{\delta}_i])_{,j}$ and $(\mathrm{Im}[\hat{\delta}_i])_{,j}$ are calculated from equation (18).

The linear increment $\Delta\gamma_1$ of γ_1 may be described as follows:

$$\Delta\gamma_1 = \left(\frac{1}{B_1}\frac{\partial B_2}{\partial \mathbf{c}} - \frac{B_2}{B_1^2}\frac{\partial B_1}{\partial \mathbf{c}}\right)\Delta\mathbf{c} = \frac{1}{B_1}\left(\frac{\partial B_2}{\partial \mathbf{c}} - \frac{\partial B_1}{\partial \mathbf{c}}\gamma_1\right)\Delta\mathbf{c} \tag{23}$$

where

$$B_1 = \left(\sum_{i=1}^{2} |\hat{\delta}_i|\right)_{,1}, \quad B_2 = \left(\sum_{i=1}^{2} |\hat{\delta}_i|\right)_{,2} \tag{24}$$

The increments $\Delta\mathbf{c}$ must satisfy the following relation due to equation (12):

$$\sum_{i=1}^{2} \Delta c_i = 0 \tag{25}$$

Equations (23) and (25) lead to the following set of simultaneous linear equations with respect to $\Delta\mathbf{c}$:

$$\begin{bmatrix} \dfrac{1}{B_1}\left(\dfrac{\partial B_2}{\partial \mathbf{c}} - \dfrac{\partial B_1}{\partial \mathbf{c}}\gamma_1\right) \\ 1 \quad 1 \end{bmatrix} \Delta\mathbf{c} = \begin{Bmatrix} \Delta\gamma_1 \\ 0 \end{Bmatrix} \tag{26}$$

Equation (26) indicates that, once $\Delta\gamma_1$ is given and $\partial B_1/\partial\mathbf{c}$, $\partial B_2/\partial\mathbf{c}$ are evaluated, $\Delta\mathbf{c}$ can be found. Let γ_{01} denote the initial value of γ_1. The increment $\Delta\gamma_1$ is given here as $\Delta\gamma_1 = (1-\gamma_{01})/N$ where N is the number of steps. It should be remarked that, if either one of c_1 or c_2 vanishes, the following relation must be satisfied. If $c_1 = 0$, $\gamma_1 \geqslant 1$. If $c_2 = 0$, $\gamma_1 \leqslant 1$.

$\partial B_1/\partial\mathbf{c}$ and $\partial B_2/\partial\mathbf{c}$ can be evaluated in the following manner. Differentiation of equation (22) with respect to c_k leads to:

$$\begin{aligned} |\hat{\delta}_i|_{,jk} = \frac{1}{|\hat{\delta}_i|^2}(|\hat{\delta}_i|\{&(\mathrm{Re}[\hat{\delta}_i])_{,k}(\mathrm{Re}[\hat{\delta}_i])_{,j} + \mathrm{Re}[\hat{\delta}_i](\mathrm{Re}[\hat{\delta}_i])_{,jk} \\ &+ (\mathrm{Im}[\hat{\delta}_i])_{,k}(\mathrm{Im}[\hat{\delta}_i])_{,j} + \mathrm{Im}[\hat{\delta}_i](\mathrm{Im}[\hat{\delta}_i])_{,jk}\} \\ &- |\hat{\delta}_i|_{,k}\{\mathrm{Re}[\hat{\delta}_i](\mathrm{Re}[\hat{\delta}_i])_{,j} + \mathrm{Im}[\hat{\delta}_i](\mathrm{Im}[\hat{\delta}_i]_{,j})\}) \end{aligned} \tag{27}$$

$\partial B_1/\partial\mathbf{c}$ and $\partial B_2/\partial\mathbf{c}$ may then be expressed as follows:

$$\frac{\partial B_1}{\partial \mathbf{c}} = \{|\hat{\delta}_1|_{,11} + |\hat{\delta}_2|_{,11} \quad |\hat{\delta}_1|_{,12} + |\hat{\delta}_2|_{,12}\} \tag{28a}$$

$$\frac{\partial B_2}{\partial \mathbf{c}} = \{|\hat{\delta}_1|_{,21} + |\hat{\delta}_2|_{,21} \quad |\hat{\delta}_1|_{,22} + |\hat{\delta}_2|_{,22}\} \tag{28b}$$

$(\mathrm{Re}[\hat{\delta}_i])_{,jk}$ and $(\mathrm{Im}[\hat{\delta}_i])_{,jk}$ in equation (27) are found from

$$\begin{Bmatrix} \hat{\delta}_{1,jk} \\ \hat{\delta}_{2,jk} \end{Bmatrix} = \mathbf{T}\mathbf{A}^{-1}\mathbf{A}_{,k}\mathbf{A}^{-1}\mathbf{A}_{,j}\mathbf{T}^{-1}\begin{Bmatrix} \hat{\delta}_1 \\ \hat{\delta}_2 \end{Bmatrix} - \mathbf{T}\mathbf{A}^{-1}\mathbf{A}_{,j}\mathbf{T}^{-1}\begin{Bmatrix} \hat{\delta}_{1,k} \\ \hat{\delta}_{2,k} \end{Bmatrix} \tag{29}$$

which is derived by differentiating equation (18) with respect to c_k and using the relation $\mathbf{A}^{-1}{}_{,k} = \mathbf{A}^{-1}\mathbf{A}_{,k}\mathbf{A}^{-1}$. It should be noted here that, since the components in the matrix $\mathbf{A}$ are linear functions of $\mathbf{c}$, $\mathbf{A}_{,jk}$ becomes a null matrix for all j and k.

3. GENERALIZATION

Consider an n-storey shear building model. The design variables are $\{c_1 \cdots c_n\}$. The problem of optimal damper placement may be stated almost in the same manner as in Problem PODP by replacing 2 by n.

Equation (2) may be generalized into the following from:

$$(\mathbf{K} + \mathrm{i}\omega\mathbf{C} - \omega^2\mathbf{M})\mathbf{U}(\omega) = -\mathbf{M}\mathbf{r}\ddot{U}_g(\omega), \tag{30}$$

where $\mathbf{K}$, $\mathbf{C}$ and $\mathbf{M}$ are the system stiffness, damping and mass matrices, respectively, and $\mathbf{r} = \{1 \cdots 1\}^{\mathrm{T}}$ is the influence coefficient vector for a base input. Furthermore, $\mathbf{U}(\omega) = \{U_1(\omega) \cdots U_n(\omega)\}^{\mathrm{T}}$.

The optimality criteria are now expressed as follows. If $c_1 \neq 0$ and $c_j \neq 0$ $(j \neq 1)$, $\gamma_{j-1} = 1$. If $c_1 \neq 0$ and $c_j = 0$ $(j \neq 1)$, $\gamma_{j-1} \leqslant 1$. If $c_1 = 0$ and $c_j \neq 0$, $\gamma_{j-1} \geqslant 1$. If $c_1 = 0$ and $c_j = 0$, γ_{j-1} is arbitrary.

Equation (6) is generalized into:

$$\mathbf{A}\hat{\mathbf{U}} = -\mathbf{Mr} \tag{31}$$

where $\hat{\mathbf{U}} = \{\hat{U}_1 \cdots \hat{U}_n\}^{\mathrm{T}}$ $(\hat{U}_j = U_j(\omega_1)/\ddot{U}_g(\omega_1))$ and

$$\mathbf{A} = \mathbf{K} + \mathrm{i}\omega_1\mathbf{C} - \omega_1^2\mathbf{M} \tag{32}$$

The transfer functions $\hat{\boldsymbol{\delta}} = \{\hat{\delta}_1 \cdots \hat{\delta}_m\}^{\mathrm{T}}$ of interstorey drifts evaluated at the undamped fundamental natural frequency are derived from $\hat{\mathbf{U}}$ by

$$\hat{\boldsymbol{\delta}} = \mathbf{T}\hat{\mathbf{U}} \tag{33}$$

where $\mathbf{T}$ is the deformation–displacement transformation matrix (generalized version of equation (4)). Equation (15) may be generalized into:

$$\mathbf{A}_{,j}\hat{\mathbf{U}} + \mathbf{A}\hat{\mathbf{U}}_{,j} = \mathbf{0} \tag{34}$$

From equation (34) $\hat{\mathbf{U}}_{,j}$ is expressed in terms of $\hat{\mathbf{U}}$.

$$\hat{\mathbf{U}}_{,j} = -\mathbf{A}^{-1}\mathbf{A}_{,j}\hat{\mathbf{U}} \tag{35}$$

It should be noted that an explicit expression of $\mathbf{A}^{-1}$ can be derived for a mass-spring model (shear building model) having a tri-diagonal matrix $\mathbf{A}$.[1,2] From equations (33) and (35) $\hat{\boldsymbol{\delta}}_{,j}$ is expressed in terms of $\hat{\boldsymbol{\delta}}$:

$$\hat{\boldsymbol{\delta}}_{,j} = -\mathbf{T}\mathbf{A}^{-1}\mathbf{A}_{,j}\mathbf{T}^{-1}\hat{\boldsymbol{\delta}} \tag{36}$$

A new quantity γ_j is defined by

$$\gamma_j = \left(\sum_{i=1}^{n} |\hat{\delta}_i|\right)_{,j+1} \Bigg/ \left(\sum_{i=1}^{n} |\hat{\delta}_i|\right)_{,1}, \quad (j = 1, \ldots, n-1) \tag{37}$$

The linear increment of γ_j may be expressed as

$$\Delta\gamma_j = \left(\frac{1}{B_1}\frac{\partial B_{j+1}}{\partial \mathbf{c}} - \frac{B_{j+1}}{B_1^2}\frac{\partial B_1}{\partial \mathbf{c}}\right)\Delta\mathbf{c} = \frac{1}{B_1}\left(\frac{\partial B_{j+1}}{\partial \mathbf{c}} - \frac{\partial B_1}{\partial \mathbf{c}}\gamma_j\right)\Delta\mathbf{c} \tag{38}$$

where

$$B_j = \left(\sum_{i=1}^{n} |\hat{\delta}_i|\right)_{,j} \tag{39}$$

Combination of equations (38) with $\sum_{i=1}^{n} \Delta c_i = 0$ yields the following set of simultaneous linear equations with respect to $\Delta\mathbf{c}$:

$$\begin{bmatrix} \dfrac{1}{B_1}\left(\dfrac{\partial B_2}{\partial \mathbf{c}} - \dfrac{\partial B_1}{\partial \mathbf{c}}\gamma_1\right) \\ \vdots \\ \dfrac{1}{B_1}\left(\dfrac{\partial B_n}{\partial \mathbf{c}} - \dfrac{\partial B_1}{\partial \mathbf{c}}\gamma_{n-1}\right) \\ 1 \cdots 1 \end{bmatrix} \Delta\mathbf{c} = \begin{Bmatrix} \Delta\gamma_1 \\ \vdots \\ \Delta\gamma_{n-1} \\ 0 \end{Bmatrix} \tag{40}$$

The derivatives $|\hat{\delta}_i|_{,jk}$ are derived from equation (27). $(\mathrm{Re}[\hat{\delta}_i])_{,j}$ and $(\mathrm{Im}[\hat{\delta}_i])_{,j}$ in equation (27) are calculated from equation (18) and $(\mathrm{Re}[\hat{\delta}_i])_{,jk}$ and $(\mathrm{Im}[\hat{\delta}_i])_{,jk}$ in equation (27) are found from equation (29).

Let $\boldsymbol{\gamma}_0$ and $\boldsymbol{\gamma}_{\mathrm{F}}$ denote the initial values for the quantities defined in equation (37) and their target values. It should be noted that $\boldsymbol{\gamma}_{\mathrm{F}} = \{1 \cdots 1\}^{\mathrm{T}}$. The linear increments $\Delta\boldsymbol{\gamma} = \{\Delta\gamma_1 \cdots \Delta\gamma_{n-1}\}^{\mathrm{T}}$ of $\boldsymbol{\gamma}$ to be specified are

given as follows:

$$\Delta\gamma = \frac{1}{N}(\gamma_{\mathrm{F}} - \gamma_0) \tag{41}$$

The solution algorithm may be summarized as follows:

Step 1: If $c_j > 0$ for all j, compute $\Delta\mathbf{c}$ from equation (40) using equation (41).
Step 2: If one of c_j's vanishes, check whether the ratio γ_{j-1} corresponding to $c_j = 0$ satisfies the condition $\gamma_{j-1} \leqslant 1$.
Step 3: Update γ_0 for the model in Step 2 and compute $\Delta\gamma$ by equation (41).
Step 4: Remove the jth column in the coefficient matrix in the left-hand side in equation (40) and the $(j-1)$th row in the same coefficient matrix corresponding to $c_j = 0$. The other Δc_j's (and c_j's) are then computed sequentially from the resulting reduced set of simultaneous linear equations.
Step 5: Repeat Steps 2–4 until all the optimality criteria are satisfied.

If $c_1 = 0$, the denominator of equation (37) should be changed to B_j for another storey, e.g. the storey with the maximum B_j value.

Since the present solution algorithm does not include any indefinite iterative operation, it will provide a definite target design (optimal design).

4. NUMERICAL EXAMPLES

4.1. Example 1 (model with a uniform distribution of storey stiffnesses)

Consider a six-degree-of-freedom model as shown in Figure 3. All the masses are assumed to be prescribed as $\bar{m}_1 = \cdots = \bar{m}_6 = 0{\cdot}8 \times 10^5$ kg. Every storey stiffness is to have the same prescribed value

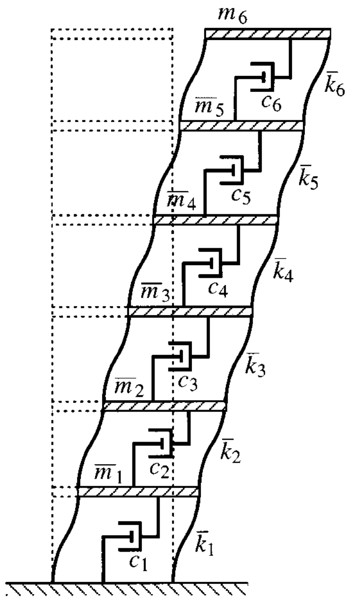

Figure 3. Six-storey shear building model with added viscous dampers

Table I. Initial values of $\{\gamma_j\}$ defined in equation (37)

j	1	2	3	4	5
γ_{0j}	0·8795	0·6798	0·4438	0·2222	0·0627

Table II. Final values of $\{\gamma_j\}$ attained in the optimal design

j	1	2	3	4	5
γ_{Fj}	1·001	0·8548	0·5550	0·2726	0·0723

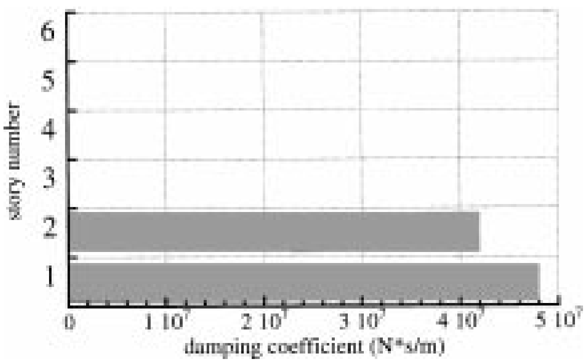

Figure 4. Distribution of optimal damping coefficients for the model with uniform distributions of masses and storey stiffnesses

$\bar{k}_1 = \cdots = \bar{k}_6 = 4{\cdot}0 \times 10^7$ (N/m). The design variables are the damping coefficients $c_1, \ldots, c_6$ of added viscous dampers. Every viscous damper is to have the same initial damping coefficient $1{\cdot}5 \times 10^6$ (Ns/m). The undamped fundamental natural circular frequency of the model is $\omega_1 = 5{\cdot}39$ (rad/s) and the initial values of γ are shown in Table I. The target values of γ are $\gamma_{Fj} = 1$ (for all j) and the final values of γ attained in the optimal design are shown in Table II. The number of steps in the redesign process is $N = 50$.

The distribution of the optimal damping coefficients obtained via the present procedure is plotted in Figure 4. It can be observed that in the optimal design the dampers are concentrated to the storyes where the largest interstorey drifts are attained in the initial design (uniform distribution of damping coefficients) (see Figure 5). This fact just corresponds to the conclusions in the previous studies.[7,11] It has been disclosed automatically through the solution algorithm explained in the previous section that unnecessary added dampers should be removed downward from the top.

Figure 5 shows the amplitudes of the transfer functions for the initial design and the optimal design. It can be observed that the amplitudes of transfer functions of interstorey drifts have been reduced significantly especially in the lower storyes. The objective function V has been reduced from 0·2139 (initial design) to 0·1351 (optimal design).

4.2. Example 2 (model with a uniform distribution of amplitudes of transfer functions)

Another shear building model with a different distribution of storey stiffnesses is considered as the second example. The distributions of masses and initial damping coefficients of added viscous dampers are the same as in the previous example. The storey stiffnesses of the shear building model have been determined so that it

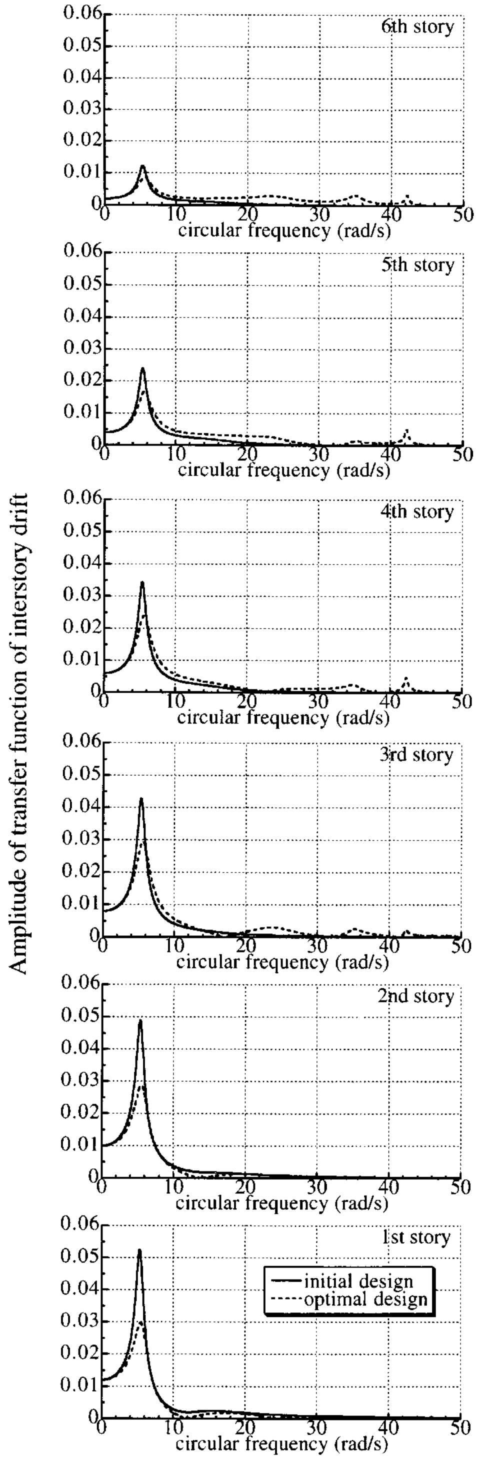

Figure 5. Amplitudes $|\delta_i(\omega)/\ddot{U}_g(\omega)|$ of transfer functions of interstorey drifts for the initial design and the optimal design

Table III. Storey stiffnesses in the second example

j	1	2	3	4	5	6
$\bar{k}_j(\times 10^7$ N/m)	5·131	4·810	4·260	3·476	2·444	1·100

Table IV. Initial values of $\{\gamma_j\}$ in the second example

j	1	2	3	4	5
γ_{0j}	1·021	1·042	1·062	1·071	0·8999

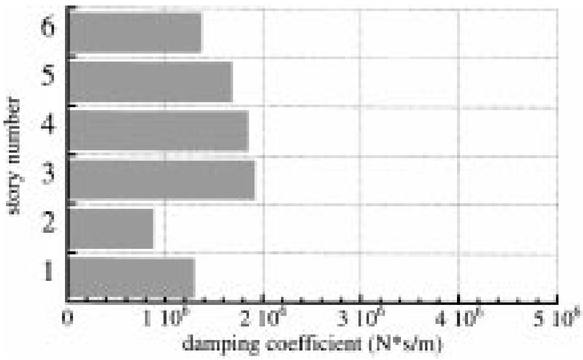

Figure 6. Distribution of optimal damping coefficients for the model with varied storey stiffnesses ($|\hat{\delta}_i|$ are uniform in the initial design)

has the undamped fundamental natural circular frequency $\omega_1 = 5{\cdot}39$ (rad/s) and the uniform distribution of the amplitudes $\{|\hat{\delta}_j|\}$ of transfer functions of interstorey drifts. This procedure can be done efficiently using an algorithm for an incremental inverse problem.[4,15] The distribution of storey stiffnesses is shown in Table III and the initial values of γ are shown in Table IV. For this model the proposed solution algorithm has been applied. The number of steps in the redesign process is $N = 500$. Figure 6 shows the distribution of the optimal damping coefficients. Different from the previous example with a uniform distribution of storey stiffnesses, the dampers are not concentrated to the specific storeys. This may result from the fact that the initial values of γ in Table IV indicate nearly optimal values in this model and drastic reduction of the objective function cannot be expected by the redistribution of added dampers. Actually, the objective function has been slightly reduced from 0·2033 to 0·2027.

5. CONCLUSIONS

A new efficient and systematic procedure has been proposed for finding the optimal damper placement in structural systems with an arbitrary damping system (for example, proportional or non-proportional). This problem is aimed at minimizing the sum of amplitudes of the transfer functions evaluated at the undamped fundamental natural frequency of a structural system subject to a constraint on the sum of the damping coefficients of added dampers. The optimal damper placement is determined based upon the newly derived optimality criteria. The features of the present formulation are to be able to deal with any damping system stated above, to be able to treat any structural system so far as it can be modelled with finite-element systems

and to consist of a systematic algorithm without any indefinite iterative operation. It is also interesting to point out that, due to employment of a general dynamical property, i.e. the amplitude of a transfer function, the results are general and are not influenced by characteristics of input motions. Efficiency and reliability of the present procedure have been demonstrated through two examples of a six-storey shear building model.

In the present paper, an upper bound of the damping coefficient of the added dampers in each storey is not given as a design constraint. However that upper bound may be necessary because the maximum power of a damper and the number of dampers that can be added to each floor are limited in a realistic situation. In this case the optimal conditions, i.e., equation (11), must be modified. The present formulation is expected to be applicable to such a case almost in the same manner. It should also be pointed out that, when optimizing damper placement, the non-stationary nature of earthquake ground motions may lead to the results different from those obtained on the basis of the assumption of stationary excitations.

The proposed technique is general and is expected to be applicable to other structural systems.

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